UG-FT-162/04
CAFPE-32/04
February 1, 2008

Recursive numerical calculus of one-loop tensor integrals

F. del Aguila and R. Pittau [1]

Departamento de Física Teórica y del Cosmos
and Centro Andaluz de Física de Partículas Elementales (CAFPE),
Universidad de Granada, E-18071 Granada, Spain

Abstract

A numerical approach to compute tensor integrals in one-loop calculations is presented. The algorithm is based on a recursion relation which allows to express high rank tensor integrals as a function of lower rank ones. At each level of iteration only inverse square roots of Gram determinants appear. For the phase-space regions where Gram determinants are so small that numerical problems are expected, we give general prescriptions on how to construct reliable approximations to the exact result without performing Taylor expansions. Working in $4+\epsilon$ dimensions does not require an analytic separation of ultraviolet and infrared/collinear divergences, and, apart from trivial integrals that we compute explicitly, no additional ones besides the standard set of scalar one-loop integrals are needed.

Keywords: One-loop Tensor integrals, Radiative Corrections, NLO Computations, Colliders

PACS Classification: 11.10.-z; 11.15.Bt; 12.38.Bx

[1] On leave of absence from Dipartimento di Fisica Teorica, Torino and INFN Sezione di Torino, Italy.

1 Introduction

Computing radiative corrections in high-energy Particle Physics demands an increasing capability of manipulating complicated objects. When the number of particles undergoing the scattering process grows, formidable complications arise. For example, to date no complete QCD or Electroweak (EW) calculation exists at the one-loop level describing processes involving 2 ingoing and 4 outgoing particles. On the other hand, multi-particle processes need to be studied with great accuracy at the next generation of pp and e^+e^- colliders, hence we cannot escape the technical subject of efficiently compute radiative corrections, at least at the one-loop level. To do this three main problems must be faced: the large number of Feynman diagrams, the reduction of tensor integrals to scalar ones, and the control over the numerical inaccuracies.

As for the first issue, while several tree level algorithms to compute amplitudes without making explicit reference to Feynman diagrams exist [1], so far there is no equivalent working technique when loops enter the game. At any rate, for moderate values of external particles the number of contributing Feynman diagrams can be in principle manageable. For example, this number is of the order of thousand for the EW process $e^+e^- \to \mu^-\bar{\nu}_\mu u\bar{d}$, depending on the chosen gauge. What really renders the calculation difficult is the fact that each diagram still requires a lot of work to be computed. This is the second issue mentioned above.

In order to be concrete, let us sketch out a typical one-loop calculation performed in $n = 4+\epsilon$ dimensions. The corresponding amplitude $A^{(n)}$ can be written as a sum of tensor integrals $I^{(n)}$ times external tensors S.

$$A^{(n)} = \sum_{m,j,i} I^{(n)}_{m,j;\,\mu_1\cdots\mu_i}\; S^{\mu_1\cdots\mu_i}_{m,j}\,, \tag{1}$$

where

$$I^{(n)}_{m,j;\,\mu_1\cdots\mu_i} = \int d^nq \frac{q_{\mu_1}\cdots q_{\mu_i}}{D_0 D_1\cdots D_m}\,,\quad D_k = (q+p_k)^2 - m_k^2\,,\quad k = 0,\cdots,m\,,\quad p_0^\mu = 0\,, \tag{2}$$

with j labelling the different momenta p_k entering and masses m_k running the loop, and where $S^{\mu_1\cdots\mu_i}_{m,j}$ only depend on the external kinematics. The $(m+1)$-point tensor one-loop integrals are usually evaluated in two steps. First one writes down the most general decomposition in terms of the external momenta and the metric tensor

$$I^{(n)}_{m,j;\,\mu_1\cdots\mu_i} = \sum_{\substack{s,\,s_1,\,\cdots,\,s_m \\ 2s+\sum s_k = i}} c^{(n)}_j\,(s,s_1,\cdots,s_m)\;\{[g]^s[p_1]^{s_1}\cdots[p_m]^{s_m}\}_{\mu_1\cdots\mu_i}\,, \tag{3}$$

where $\{[g]^s[p_1]^{s_1}\cdots[p_m]^{s_m}\}_{\mu_1\cdots\mu_i}$ corresponds to the symmetrical tensor combination, each term of which is constructed from s metric tensors, s_1 momenta p_1, $\cdots$, and s_m momenta p_m. For example,

$$\{gp_1\}_{\mu_1\mu_2\mu_3} = g_{\mu_1\mu_2}p_{1_{\mu_3}} + g_{\mu_1\mu_3}p_{1_{\mu_2}} + g_{\mu_2\mu_3}p_{1_{\mu_1}}\,. \tag{4}$$

Secondly one looks for explicit expressions for the scalar coefficients $c_j^{(n)}(s, s_1, \cdots, s_m)$. In the Passarino-Veltman treatment [2] they are expressed in terms of a minimal set of scalar one-loop integrals with denominators raised to the first power only [3]

$$I_{m,j}^{(n)} = \int d^n q \frac{1}{D_0 D_1 \cdots D_m} . \tag{5}$$

In other methods [4, 5] the decomposition is performed in terms of an enlarged set of scalar integrals with shifted dimensionality and denominators raised to generic powers

$$I_{m,j,\nu_j}^{(n+\nu)} = \int d^{(n+\nu)} q \frac{1}{D_0^{\nu_0} D_1^{\nu_1} \cdots D_m^{\nu_m}} . \tag{6}$$

Of course, there are relations among the integrals in Eq. (6) and the minimal set in Eq. (5), which have been extensively studied in Ref. [4]. For large m values the algebraical complexity of the described methods quickly becomes overwhelming, mainly due to the rapidly increasing number of kinematic variables present in the problem. For example, the application of Eq. (3) to $I^{(n)}_{4,j;\,\mu_1\mu_2\mu_3}$ generates 24 scalar coefficients.

An alternative to this procedure is a numerical approach. The ideal situation would be that one simply writes down Eq. (1) while all the rest is handled by a numerical program. The main problem with such an approach is handling ultraviolet and infrared/collinear singularities. Along this road a complete formalism has been recently presented in the framework of QCD by Giele and Glover [6]. In their method one first analytically separates the divergent contributions arising from the tensor integrals in Eq. (2), using the techniques of Refs. [7, 8, 9]. Then one computes numerically the kinematical coefficients of the resulting finite 4-dimensional integrals. A different approach is sewing tree amplitudes together to construct loop amplitudes, as proposed in Ref. [10]. Another way is combining virtual and real contributions to cancel the divergences in the loop integration [11], or constructing counter-terms diagram by diagram [12]. Finally, the authors of Ref. [13] develop a pure numerical approach where in the Feynman parameter space any one-loop integrand is cast in a form well suited for numerical calculation, in such a way that all possible divergences get automatically extracted.

In this paper we present a new method in which almost all the work can be performed numerically: the tensor integrals in Eq. (2) are numerically reduced to the minimal set in Eq. (5), and the ultraviolet and infrared/collinear divergences are controlled without performing any explicit subtraction. Then, any amplitude can be calculated simply contracting the numerically computed tensor integrals with the external tensors $S_{m,j}$ (see Eq. (1)). Our main result will be the derivation of the set of recursion relations that link high rank tensor integrals to lower rank ones, and which allow the numerical reduction of the former to the minimal set in Eq. (5). On the other hand, our solution to the problem of handling divergences is splitting beforehand any tensor into its pure 4-dimensional part plus any other additional contributions, which are trivial to compute. Such a procedure renders unnecessary an analytical separation of the divergent parts, as required by the recursion relations of Ref. [7], and minimizes the analytic work. At the end all divergences are contained in the pole parts of the scalar integrals, parts to which one can give any value to numerically check all relevant cancellations. Furthermore, internal and/or external masses do not pose any particular problem, so that the method can be applied to both QCD and EW calculations.

The tensor integrals are usually well behaved when two or more momenta become linearly dependent, as observed in Refs. [14, 15, 16]. However, in general reduction formulae introduce inverse powers of Gram determinants

$$\Delta_{1\cdots m} = \mathrm{Det}\,(p_i \cdot p_j)\,, \qquad 1 \leq i, j \leq m \tag{7}$$

in the decomposition terms. So when they go to zero, large cancellations among the different terms must take place, giving rise to numerical instabilities. This is the third problem one has to face. In Ref. [17] this is solved by systematically building up combinations of well behaved functions in the limit of vanishing Gram determinants. The drawback of this approach is that the class of needed loop functions has to be enlarged, including integrals computed in higher dimensions, such as those in Eq. (6). When things are re-expressed in terms of $(4+\epsilon)$-dimensional integrals, Gram determinants are reintroduced and explicit Taylor expansions are required to deal with the problematic phase-space regions. A possible solution is presented in Ref. [18], where an extrapolation from the inner phase-space region is used for the *dangerous* points. In the numerical approach of Ref. [13] all scalar coefficients $c^{(n)}$ in Eq. (3) are cast in a form well suited for numerical evaluation. In Ref. [19] 4-dimensional pentagon like tensors are reduced to box like tensors, avoiding the occurrence of rank $m = 4$ Gram determinants. Finally, the case of *exactly* zero Gram determinant is solved in Ref. [7] using the pseudo-inverse of the Gram matrix.

In our method part of the Gram determinant singularities compensates in such a way that at each step of the iteration only inverse square roots of Gram determinants appear, improving the numerical stability of the calculation. Moreover, the expressions can be very naturally arranged in groups which are well behaved in the limit of linearly dependent external momenta, allowing to keep as local as possible the numerical cancellations that occur among the loop functions. The case of *exactly* zero Gram determinants can be treated without any problem, and for the problematic regions of nearly vanishing Gram determinants, we show how to systematically construct reliable approximations to the exact result. Unlike in Ref. [17] building up such approximations does not require explicit Taylor expansions.

The paper is organised as follows. Our master formula for the general 4-dimensional case is presented in next Section. The algorithm is extended to the n-dimensional case in Section 3. The particular cases of 3-point and rank one tensors, to which our general formula does not apply, are discussed in Sections 4 and 5, respectively. Whereas in Section 6 we perform a detailed study of the dangerous collinear and coplanar configurations. Section 7 is devoted to conclusions; and technical details are worked out in three Appendices.

2 The 4-dimensional method

In the following we shall show our master formula to reduce high rank tensor integrals to lower rank ones. Let us first clarify the notation. Throughout the paper we drop the index (n) from loop integrals when the space-time is 4-dimensional. We also omit the index j for simplicity, although it is understood that an $(m+1)$-point integral depends on m external momenta and $m+1$ internal masses. m-point tensor integrals obtained from an $(m+1)$-point one by eliminating the k^{th} denominator are written specifying the index of the dropped denominator as an argument

$$I_{m-1;\,\mu\cdots\nu}\,(k) = \int d^4q \frac{q_\mu \cdots q_\nu}{D_0 D_1 \cdots D_{k-1} D_{k+1} \cdots D_m}\,; \tag{8}$$

and as usual tensor indices are raised and lowered with the metric tensor.

Our main result is the recursion relation for reducing 4-dimensional tensor loop integrals with $m > 2$ and rank higher than 1,

$$\begin{aligned} I_{m;\,\mu\nu\rho\cdots\tau} &= \frac{\beta}{2\gamma} T_{\mu\nu\lambda\sigma} \left\{ J^{\lambda\sigma}_{m;\,\rho\cdots\tau} \right\} - \frac{1}{4\,\gamma} T_{\mu\nu} \left\{ m_0^2 I_{m;\,\rho\cdots\tau} + I_{m-1;\,\rho\cdots\tau}(0) \right\} \\ &- \frac{1}{4\,\gamma} T_{\mu\nu\lambda} \left\{ f_{30} I^\lambda_{m;\,\rho\cdots\tau} + I^\lambda_{m-1;\,\rho\cdots\tau}(3) - I^\lambda_{m-1;\,\rho\cdots\tau}(0) - \frac{2\beta}{\gamma} p_{3\alpha}\, J^{\alpha\lambda}_{m;\,\rho\cdots\tau} \right\}\,, \end{aligned} \tag{9}$$

where

$$\begin{aligned} J^{\lambda\sigma}_{m;\,\rho\cdots\tau} &= (f_{10} r_2^\lambda + f_{20} r_1^\lambda) I^\sigma_{m;\,\rho\cdots\tau} + r_2^\lambda I^\sigma_{m-1;\,\rho\cdots\tau}(1) + r_1^\lambda I^\sigma_{m-1;\,\rho\cdots\tau}(2) \\ &- (r_1^\lambda + r_2^\lambda) I^\sigma_{m-1;\,\rho\cdots\tau}(0)\,, \end{aligned} \tag{10}$$

and $T_{\mu\nu\lambda\sigma}$, $T_{\mu\nu\lambda}$, $T_{\mu\nu}$ and $r^\lambda_{1,2}$ only depend on three linearly independent external momenta, which we assume to be $p_{1,2,3}$. The scalar factors β and γ are only functions of $p_{1,2}$; whereas

$$f_{k0} = m_k^2 - m_0^2 - p_k^2\,. \tag{11}$$

Eq. (9) can then be iterated to compute numerically all 4-dimensional tensors $I_{m,j;\,\mu_1\cdots\mu_i}$, without explicit tensor decomposition, starting from the standard set of scalar loop functions in Eq. (5). Notice that the variable shift $q \to q - p_1$ is needed before applying the next iteration to any tensor integral having (0) as an argument.

In order to derive Eq. (9), we need to express the product of two loop momenta as a sum of terms with at most one loop momentum times loop denominators, or internal masses, and external tensors (Eq. (24) below). First, we write the internal momentum as a sum of four judiciously chosen massless vectors

$$q = \sum_{i=1}^{4} c_i \ell_i\,, \qquad \ell_i^2 = 0\,. \tag{12}$$

Following Ref. [20] we construct $\ell_{1,2}$ from two independent external momenta, which we assume to be $p_{1,2}$,

$$p_1 = \ell_1 + \alpha_1 \ell_2\,, \quad p_2 = \ell_2 + \alpha_2 \ell_1\,. \tag{13}$$

Then [2]

$$
\begin{aligned}
&\ell_1 = \beta\,(p_1 - \alpha_1\, p_2), && \ell_2 = \beta\,(p_2 - \alpha_2\, p_1), \\
&\alpha_1 = ((p_1 \cdot p_2) \mp \sqrt{\Delta})/p_2^2, && \Delta \equiv -\Delta_{12} = (p_1 \cdot p_2)^2 - p_1^2 p_2^2, \\
&\alpha_2 = \alpha_1\, p_2^2/p_1^2, && \beta = 1/(1-\alpha_1\alpha_2).
\end{aligned} \tag{14}
$$

Note that $\ell_{1,2}$ can be also complex. The other two independent massless vectors $\ell_{3,4}$ are taken to be

$$
\ell_3^\mu = \bar{v}(\ell_1)\,\gamma^\mu\,\omega^-\,u(\ell_2)\,, \quad \ell_4^\mu = \bar{v}(\ell_2)\,\gamma^\mu\,\omega^-\,u(\ell_1)\,, \quad \omega^- = (1-\gamma_5)/2\,, \tag{15}
$$

thus fulfilling the equalities (see Appendix A)

$$
(\ell_{3,4} \cdot \ell_{1,2}) = \ell_3^2 = \ell_4^2 = 0\,, \quad (\ell_3 \cdot \ell_4) = -4(\ell_1 \cdot \ell_2)\,. \tag{16}
$$

Using them the coefficients c_i in Eq. (12) can be simply written

$$
c_1 = \frac{(q\cdot\ell_2)}{(\ell_1\cdot\ell_2)}\,, \quad c_2 = \frac{(q\cdot\ell_1)}{(\ell_1\cdot\ell_2)}\,, \quad c_3 = -\frac{(q\cdot\ell_4)}{4\,(\ell_1\cdot\ell_2)}\,, \quad c_4 = -\frac{(q\cdot\ell_3)}{4\,(\ell_1\cdot\ell_2)}\,. \tag{17}
$$

Now, it is convenient to distinguish between $\ell_{1,2}$ and $\ell_{3,4}$ in Eq. (12) for the contribution of the first two 4-vectors can be expressed as a sum of products of denominators, internal masses and external momenta,

$$
\begin{aligned}
q_\mu &= \frac{\beta}{\gamma} D_\mu - \frac{1}{2\,\gamma} Q_\mu\,, \\
D_\mu &= \frac{1}{\beta}\left[2\,(q\cdot\ell_1)\ell_{2\mu} + 2\,(q\cdot\ell_2)\ell_{1\mu}\right] \\
&= \left\{f_{10}\, r_{2\mu} + f_{20}\, r_{1\mu} + D_1\, r_{2\mu} + D_2\, r_{1\mu} - D_0\,(r_{1\mu} + r_{2\mu})\right\}\,, \\
Q_\mu &= (q\cdot\ell_3)\,\ell_{4\mu} + (q\cdot\ell_4)\,\ell_{3\mu}\,, \quad \gamma \;=\; 2\frac{(p_1\cdot p_2)}{1+\alpha_1\,\alpha_2} \;=\; 2\,(\ell_1\cdot\ell_2)\,,
\end{aligned} \tag{18}
$$

where we have made use of

$$
(q\cdot p_k) = \frac{1}{2}\left[D_k - D_0 + f_{k0}\right]\,, \tag{19}
$$

[2] When one or both p_i^2 vanishes, $\beta = 1$, and

$$
\begin{array}{llll}
p_1^2 = 0, \quad p_2^2 \neq 0 & \Rightarrow & \alpha_1 = 0, & \alpha_2 = p_2^2/(2\,(p_1\cdot p_2))\,, \\
p_1^2 \neq 0, \quad p_2^2 = 0 & \Rightarrow & \alpha_1 = p_1^2/(2\,(p_1\cdot p_2)), & \alpha_2 = 0\,, \\
p_1^2 = 0, \quad p_2^2 = 0 & \Rightarrow & \alpha_1 = 0, & \alpha_2 = 0\,.
\end{array}
$$

These limits are smoothly approached taking for α_1 the solution with $-\sqrt{\Delta}$ ($+\sqrt{\Delta}$) when $(p_1\cdot p_2) > 0$ ($(p_1\cdot p_2) < 0$), which is what we stand for $\mp\sqrt{\Delta}$.

with f_{k0} defined in Eq. (11), and

$$r_1 = (\ell_1 - \alpha_1\, \ell_2), \quad r_2 = (\ell_2 - \alpha_2\, \ell_1). \tag{20}$$

Then applying Eq. (18) twice and symmetrizing on $\mu\nu$, we find

$$\begin{aligned} q_\mu q_\nu &= \frac{\beta}{2\gamma}\left[D^\lambda q^\sigma\right]\left\{g_{\lambda\mu}\left(g_{\sigma\nu} - \frac{t_{\sigma\nu}}{2\gamma}\right) + g_{\lambda\nu}\left(g_{\sigma\mu} - \frac{t_{\sigma\mu}}{2\gamma}\right)\right\} + \frac{Q_\mu Q_\nu}{4\,\gamma^2}\,, \\ t_{\rho\tau} &= \ell_{3\rho}\ell_{4\tau} + \ell_{4\rho}\ell_{3\tau}\,. \end{aligned} \tag{21}$$

The factor in square brackets contains reconstructed denominators while the factor in curly brackets only depends on the external kinematics. The $Q_\mu Q_\nu$ term can be also decomposed using the properties of the 4-vectors $\ell_{3,4}$ (see Appendix A for details)

$$\begin{aligned} (q\cdot\ell_3)(q\cdot\ell_4) &= 4\,(q\cdot\ell_1)\,(q\cdot\ell_2) - 2\,q^2\,(\ell_1\cdot\ell_2) \;\equiv\; \beta q^\alpha D_\alpha - \gamma q^2\,, \\ (q\cdot\ell_{3(4)})(q\cdot\ell_{3(4)}) &= \frac{2}{(b\cdot\ell_{4(3)})}\Big\{\left[q^2(\ell_1\cdot\ell_2) - 2(q\cdot\ell_1)(q\cdot\ell_2)\right](b\cdot\ell_{3(4)}) \\ &+ 2\left[(q\cdot\ell_1)(b\cdot\ell_2) - (q\cdot b)(\ell_1\cdot\ell_2) + (q\cdot\ell_2)(\ell_1\cdot b)\right](q\cdot\ell_{3(4)})\Big\} \\ &\equiv \frac{1}{(b\cdot\ell_{4(3)})}\Big\{\left[\gamma q^2 - \beta q^\alpha D_\alpha\right](b\cdot\ell_{3(4)}) \\ &- \left[\gamma\,(q\cdot b) - \beta b^\alpha D_\alpha\right](q\cdot\ell_{3(4)})\Big\}\,, \quad \forall\, b \neq \ell_{1,2}\,. \end{aligned} \tag{22}$$

Now, choosing the arbitrary 4-vector $b = p_3$ and reconstructing again denominators, we get

$$\begin{aligned} Q_\mu Q_\nu &= \beta\left[D^\rho q^\alpha\right]\{g_{\rho\alpha}T_{\mu\nu}\} - \gamma\left[D_0 + m_0^2\right]\{T_{\mu\nu}\} \\ &+ \left[2\,\beta\,p_{3\alpha}\left[D^\alpha q^\lambda\right] - \gamma\,(D_3 - D_0 + f_{30})q^\lambda\right]\{T_{\mu\nu\lambda}\}\,, \\ T_{\mu\nu\lambda} &= \frac{\ell_{3\mu}\,\ell_{3\nu}\,\ell_{4\lambda}}{(\ell_3\cdot p_3)} + \frac{\ell_{4\mu}\,\ell_{4\nu}\,\ell_{3\lambda}}{(\ell_4\cdot p_3)}, \\ T_{\mu\nu} &= t_{\mu\nu} - p_3^\sigma T_{\mu\nu\sigma}\,. \end{aligned} \tag{23}$$

Finally, inserting Eq. (23) into Eq. (21) we obtain the decomposition relation for the product of two loop momenta

$$\begin{aligned} q_\mu q_\nu &= \frac{\beta}{2\gamma}\left[D^\lambda q^\sigma\right]T_{\mu\nu\lambda\sigma} - \frac{1}{4\,\gamma}\left[D_0 + m_0^2\right]T_{\mu\nu} \\ &- \frac{1}{4\,\gamma}\left[\left((D_3 - D_0 + f_{30}) - \frac{2\beta}{\gamma}p_{3\alpha}\,D^\alpha\right)q^\lambda\right]T_{\mu\nu\lambda}\,, \end{aligned} \tag{24}$$

where

$$T_{\mu\nu\lambda\sigma} = g_{\mu\lambda}\left(g_{\nu\sigma} - \frac{t_{\nu\sigma}}{2\gamma}\right) + g_{\nu\lambda}\left(g_{\mu\sigma} - \frac{t_{\mu\sigma}}{2\gamma}\right) + \frac{g_{\lambda\sigma}}{2\gamma}\,T_{\mu\nu}\,. \tag{25}$$

Then Eq. (9) follows after dividing Eq. (24) by the $(m+1)$ denominators in Eq. (2), multiplying by the remaining 4-vectors $q_\rho \cdots q_\tau$ and integrating over d^4q. As it stands, Eq. (9) is valid in non exceptional phase-space regions where $p_{1,2,3}$ are all independent, otherwise zeros occur in the denominators. In Section 6, we shall consider the case of exceptional or nearly exceptional configurations. The reason why it is convenient to group terms in the way we have done will be clear there.

As a last remark we observe that the rank of the tensor integral on the l.h.s. of Eq. (9) should be at least 2 and $m \geq 3$, otherwise there are not enough external momenta to perform the decomposition neither denominators to cancel the reconstructed ones. These two particular cases are discussed in Sections 5 and 4, respectively.

3 The method in n dimensions

The derivation of Eq. (9) breaks down when working in n dimensions because Eq. (12) is only valid in 4 dimensions. To deal with the n-dimensional case we need to reduce the problem to 4-dimensional tensors, that we know how to handle. As in Refs. [20, 21, 22, 23] only unobservable objects are considered to live in $n = 4 + \epsilon$ dimensions, with 4 and ϵ-dimensional quantities always orthogonal to each other. In particular, only the integration momentum q is n-dimensional in our integrals. For notational purposes from now on we put a bar over n-dimensional quantities and a tilde over ϵ-dimensional objects. For example,

$$\bar{q} = q + \tilde{q}\,, \qquad \bar{q}^2 = q^2 + \tilde{q}^2\,, \tag{26}$$

where q is purely 4-dimensional. Being more explicit, the rank i n-dimensional tensor integrals we want to evaluate read

$$I^{(n)}_{m;\,\bar{\mu}_1\cdots\bar{\mu}_i} = \int d^n\bar{q}\,\frac{1}{\bar{D}_0\cdots\bar{D}_m}\prod_{k=1}^{i}\bar{q}_{\mu_k}\,, \qquad \bar{D}_k = (\bar{q}+p_k)^2 - m_k^2\,. \tag{27}$$

Using Eq. (26) to split the numerator momenta of Eq. (27), we get

$$\begin{aligned}\prod_{k=1}^{i}\bar{q}_{\mu_k} &= \prod_{k=1}^{i} q_{\mu_k} + \sum_{s_1=1}^{i}\tilde{q}_{\mu_{s_1}}\prod_{k\neq s_1}^{i} q_{\mu_k} \\ &+ \sum_{s_1>s_2}^{i}\tilde{q}_{\mu_{s_1}}\tilde{q}_{\mu_{s_2}}\prod_{k\neq s_1,s_2}^{i} q_{\mu_k} + \cdots + \prod_{k=1}^{i}\tilde{q}_{\mu_k}\,.\end{aligned} \tag{28}$$

Since the momenta p_k are purely 4-dimensional, all terms containing an odd number of $\tilde{q}$ in the numerator vanish, and all terms with an even number of $\tilde{q}$ can be only proportional to symmetric combinations of the metric tensor in ϵ dimensions

$$\tilde{g}_{\mu\nu} \equiv g_{\tilde{\mu}\tilde{\nu}}\,, \quad \tilde{g}_{\mu\nu\rho\sigma} \equiv \tilde{g}_{\mu\nu}\,\tilde{g}_{\rho\sigma} + \tilde{g}_{\mu\rho}\,\tilde{g}_{\nu\sigma} + \tilde{g}_{\mu\sigma}\,\tilde{g}_{\nu\rho}\,, \quad \text{etc.} \tag{29}$$

Therefore

$$I^{(n)}_{m;\,\mu_1\cdots\mu_h\tilde{\nu}_1\cdots\tilde{\nu}_{2\ell+1}} \equiv \int d^n\bar{q}\,\frac{1}{\bar{D}_0\cdots\bar{D}_m}\prod_{k=1}^{h} q_{\mu_k}\prod_{s=1}^{2\ell+1}\tilde{q}_{\nu_s} = 0\,, \tag{30}$$

and

$$I^{(n)}_{m;\,\mu_1\cdots\mu_h\tilde{\nu}_1\cdots\tilde{\nu}_{2\ell}} = \Gamma\left(\frac{\epsilon}{2}\right)\frac{\tilde{g}_{\nu_1\cdots\nu_{2\ell}}}{2^\ell\,\Gamma\left(\frac{\epsilon}{2}+\ell\right)}\int d^n\bar{q}\frac{\tilde{q}^{2\ell}}{\bar{D}_0\cdots\bar{D}_m}\prod_{k=1}^{h} q_{\mu_k}\,, \tag{31}$$

where $\tilde{q}^2 = \tilde{q}_\mu\tilde{q}^\mu$ and $\ell = 0, 1, 2, \cdots$. As it is clear from the previous equation, a new class of integrals involving $\tilde{q}^{2\ell}$ appears. We will use for these objects the notation $I^{(n;\,2\ell)}$. For example, the last integral in Eq. (31) is denoted by $I^{(n;\,2\ell)}_{m;\,\mu_1\cdots\mu_h}$. Such integrals are very easy to evaluate at $\mathcal{O}(1)$, and we do it in Appendix B. One can argue that, since a pole in $1/\epsilon$ appears in Eq. (31), higher order terms might be needed. However, the ϵ-dimensional tensor indices in Eq. (31) can only be contracted with ϵ-dimensional indices (typically combinations of $\tilde{g}_{\mu\nu}$ tensors), otherwise they give zero because of the orthogonality between 4-dimensional and ϵ-dimensional spaces. In this contraction the pole cancels out, so that $\mathcal{O}(\epsilon)$ terms can be safely neglected in the physical limit $\epsilon \to 0$.

Finally, the purely 4-dimensional integral $I^{(n)}_{m;\,\mu_1\cdots\mu_i}$, coming from the first term of Eq. (28), can be reduced using Eq. (9) with $I \to I^{(n)}$ [3]. However, there is still one important modification we have to take care of. The 4-dimensional denominators appearing in Eq. (24) differ from the n-dimensional ones by an amount $\tilde{q}^2$

$$D_k = \bar{D}_k - \tilde{q}^2\,. \tag{32}$$

To compensate for this, the only replacement needed in Eq. (9), before applying it to the n-dimensional case, is

$$m_0^2 I_{m;\rho\cdots\tau} \to m_0^2 I^{(n)}_{m;\rho\cdots\tau} - I^{(n;\,2)}_{m;\rho\cdots\tau} \tag{33}$$

Summarizing, any n-dimensional one-loop amplitude can be written

$$A^{(n)} = \sum_{m,j,i} I^{(n)}_{m,j;\,\bar{\mu}_1\cdots\bar{\mu}_i}\;S^{\bar{\mu}_1\cdots\bar{\mu}_i}_{m,j}\,. \tag{34}$$

After splitting the n-dimensional momenta according to Eq. (28), $I^{(n)}_{m,j;\,\mu_1\cdots\mu_i}$ can be evaluated with the help of Eq. (9), together with the replacement given in Eq. (33). The additional integrals are always of the type $I^{(n;\,2\ell)}_{m;\,\mu_1\cdots\mu_h}$. Since, after all, they are also 4-dimensional tensors, it would be in principle possible to compute them with the help of Eq. (9). In practice, the direct computation given in Appendix B is more convenient when one is interested in taking the limit $\epsilon \to 0$.

To conclude, we notice that the technique of splitting loop tensors in 4-dimensional plus ϵ-dimensional parts can help also outside the algorithm we are presenting here. For example, the method to reduce pentagon tensor integrals to box tensor ones presented in Ref. [19] relies on 4-dimensional objects. However, a strategy for a possible extension to n-dimensions, which requires an explicit subtraction of soft and collinear divergences [18, 24], is outlined by the authors. Instead, as described above, via the splitting in Eq. (28), the ϵ-dimensional

[3] Of course, the tensors in the numerator of $I^{(n)}$ stay 4-dimensional.

part of the tensors can be computed separately; and the algorithm of Ref. [19] directly applied to the remaining 4-dimensional part of the tensor integrals ($I^{(n)}_{4;\,\mu_1\cdots\mu_i}$ in our notation) without any need of introducing a regulator Λ in the intermediate stages of the calculation. The only additional modification is the replacement in Eq. (32) for all the reconstructed denominators appearing in Ref. [19].

4 Three-point tensors

Eq. (9) cannot be applied when $m = 2$ because of the lack of a third 4-momentum p_3 to reconstruct denominators. In this Section we derive a specific recursion relation for this case, which is valid for rank 2 and rank 3 three-point tensor integrals: [4]

$$I^{(n)}_{2;\,\mu\nu(\rho)} = \frac{\beta}{2\gamma} T'_{\mu\nu\lambda\sigma} \left\{ J^{(n)\,\lambda\sigma}_{2;\,(\rho)} \right\} - \frac{1}{4\,\gamma} t_{\mu\nu} \left\{ m_0^2\, I^{(n)}_{2;\,(\rho)} + I^{(n)}_{1;\,(\rho)}(0) - I^{(n;\,2)}_{2;\,(\rho)} \right\} , \qquad (35)$$

where $J^{(n)\,\lambda\sigma}_{2;\,(\rho)}$ is the combination of one-loop integrals given in Eq. (10) and

$$T'_{\mu\nu\lambda\sigma} = g_{\mu\lambda} \left(g_{\nu\sigma} - \frac{t_{\nu\sigma}}{2\gamma} \right) + g_{\nu\lambda} \left(g_{\mu\sigma} - \frac{t_{\mu\sigma}}{2\gamma} \right) + \frac{g_{\lambda\sigma}}{2\gamma}\, t_{\mu\nu} \,. \qquad (36)$$

Things get more complicated for higher rank tensors and the general solution is given in Appendix C.

To derive the former recursion relation, we shall make use of the following Theorem:

$$\int d^n\bar{q} \frac{1}{\bar{D}_0 \bar{D}_1 \bar{D}_2} (q\cdot\ell_3)^i = 0 \,,$$
$$\int d^n\bar{q} \frac{1}{\bar{D}_0 \bar{D}_1 \bar{D}_2} (q\cdot\ell_4)^i = 0 \,, \qquad \forall\, i = 1, 2, 3 \cdots \,. \qquad (37)$$

Proof:

$$\int d^n\bar{q} \frac{1}{\bar{D}_0 \bar{D}_1 \bar{D}_2} (q\cdot\ell_3)^i = \ell_3^{\mu_1} \cdots \ell_3^{\mu_i}\, I^{(n)}_{2;\mu_1\cdots\mu_i} \,. \qquad (38)$$

On the other hand, the tensor integral $I^{(n)}_{2;\mu_1\cdots\mu_i}$ admits a decomposition in terms of momenta $p_{1,2}$ and metric tensors

$$\cdots p_{1\mu_k} \cdots p_{2\mu_j} \cdots g_{\mu_\ell \mu_h} \cdots \,. \qquad (39)$$

Then, as $(\ell_3 \cdot p_{1,2}) = 0$, all tensor structures containing $p_{1\mu_k}$ or $p_{2\mu_j}$ will vanish when contracted with $\ell_3^{\mu_k}$ or $\ell_3^{\mu_j}$. Analogously, $g_{\mu_\ell\mu_h}$ cancels when contracted with $\ell_3^{\mu_\ell}\ell_3^{\mu_h}$ because $\ell_3^2 = 0$. What proves the theorem. In the same way it can be shown the identity for ℓ_4.

[4] The third index is in parentheses to remind that the equation is valid for rank 2 and 3.

Corollary:

$$\int d^n\bar{q}\frac{1}{\bar{D}_0\bar{D}_1\bar{D}_2}(q\cdot\ell_3)^2\,q_\rho \;=\; \int d^n\bar{q}\frac{1}{\bar{D}_0\bar{D}_1\bar{D}_2}(q\cdot\ell_4)^2\,q_\rho \;=\; 0\,. \tag{40}$$

It can be proved again performing an explicit tensor decomposition. This theorem allows to replace in the integrand of a rank 2 or 3 three-point integral

$$Q_\mu Q_\nu\,(q_\rho) \;\to\; \beta\left[D^\lambda q^\alpha\right]\{g_{\lambda\alpha}t_{\mu\nu}\}\;(q_\rho) - \gamma\,q^2\,\{t_{\mu\nu}\}\;(q_\rho)\,. \tag{41}$$

Indeed, $Q_\mu Q_\nu$ in Eq. (21) read

$$Q_\mu Q_\nu = (q\cdot\ell_3)^2\ell_{4\mu}\ell_{4\nu} + (q\cdot\ell_4)^2\ell_{3\mu}\ell_{3\nu} + (q\cdot\ell_3)(q\cdot\ell_4)t_{\mu\nu}\,. \tag{42}$$

Then, the previous theorem guarantees that the first two terms give zero after integration, and the corollary that the same remains true when they are multiplied by *one and only one* additional integration momentum q_ρ. Finally, Eq. (41) and the same steps as in Sections 2 and 3 result in the recursion relation (35).

To conclude, we observe that similar methods can be also used to compute two-point tensors, as explicitly shown, for a particular case, at the beginning of next Section. However, the case $m = 1$ is so simple that we do not find any advantage with respect to standard reduction techniques.

5 Rank one tensors

As already observed at the end of Section 2, Eq. (9) cannot be applied, as it stands, to reduce tensors of rank one. In this Section we show how to cope with this situation.

5.1 The $m = 1$ case

The standard Passarino-Veltman decomposition gives

$$I^{(n)}_{1;\,\mu} = \frac{p_1^\mu}{2\,p_1^2}\left\{f_{10}\,I_1^{(n)} + I_0^{(n)}(1) - I_0^{(n)}(0)\right\}\,. \tag{43}$$

The same result can be derived extending our method. We can write

$$p_1 = \ell_1 + \frac{\ell_2}{2} \;\text{ with }\; \ell_{1,2}^2 = 0 \;\text{ and }\; (\ell_1\cdot\ell_2) = 2\,(\ell_1\cdot p_1) = (\ell_2\cdot p_1) = p_1^2\,, \tag{44}$$

what corresponds to the second case of Footnote 2 with $\alpha_1 = 1/2$. The massless 4-vectors $\ell_{1,2}$ in Eq. (44) always exist. For example, if $p_1^2 > 0$, $\ell_1 = (M/2, -M/2, 0, 0)$ and $\ell_2 = (M, M, 0, 0)$ in the frame where $p_1 = (M, \vec{0})$. The corresponding 4-vectors $\ell_{3,4}$ are defined as in Eq. (15). As besides $I^{(n)}_{1;\,\mu}$ must be proportional to $p_{1\mu}$, Eqs. (16) and (44) imply

$$\begin{aligned}\int d^n\bar{q}\frac{1}{\bar{D}_0\bar{D}_1}(q\cdot\ell_{3,4}) &= 0\,,\\ \int d^n\bar{q}\frac{1}{\bar{D}_0\bar{D}_1}\,2\,(q\cdot\ell_1) &= \int d^n\bar{q}\frac{1}{\bar{D}_0\bar{D}_1}(q\cdot\ell_2)\\ &= \int d^n\bar{q}\frac{1}{\bar{D}_0\bar{D}_1}(q\cdot p_1)\,. \end{aligned} \tag{45}$$

Using them and Eq. (18) one obtains

$$\int d^n\bar{q}\,\frac{q^\mu}{\bar{D}_0\bar{D}_1} = \frac{1}{\gamma}\int d^n\bar{q}\,\frac{2\,(q\cdot p_1)\,p_1^\mu}{\bar{D}_0\bar{D}_1}\,, \tag{46}$$

and Eq. (43).

5.2 The $m = 2$ case

Using Eq. (18) and the Theorem in Eq. (37) one can show that

$$\int d^n\bar{q}\,\frac{q^\mu}{\bar{D}_0\bar{D}_1\bar{D}_2} = \frac{\beta}{\gamma}\int d^n\bar{q}\,\frac{D^\mu}{\bar{D}_0\bar{D}_1\bar{D}_2}\,; \tag{47}$$

so that

$$I^{(n)}_{2;\,\mu} = \frac{\beta}{\gamma}J^{(n)}_{2;\,\mu}\,, \tag{48}$$

where J is defined in Eq. (10).

5.3 The $m = 3$ case

We use again Eq. (18) to write

$$I^{(n)}_{3;\,\mu} = \int\frac{d^n\bar{q}}{\bar{D}_0\bar{D}_1\bar{D}_2\bar{D}_3}\left[\frac{\beta\,D_\mu}{\gamma} - \frac{\ell_{3\mu}(q\cdot\ell_4)+\ell_{4\mu}(q\cdot\ell_3)}{2\,\gamma}\right]. \tag{49}$$

Multiplying and dividing $(q\cdot\ell_{3,4})$ by $(p_3\cdot\ell_{4,3})$ we can express

$$(q\cdot\ell_{3,4}) = \frac{\mathrm{Tr}[\not{\ell}_1\not{q}\not{\ell}_2\not{p}_3\,\omega^-]}{(p_3\cdot\ell_{4,3})}\,, \tag{50}$$

where the trace containing γ_5 vanishes upon integration because it is proportional to the totally antisymmetric tensor $\epsilon(\ell_1, q, \ell_2, p_3)$. Then, the following substitution is allowed in the integrand of Eq. (49) (where we also use Eq. (18))

$$(q\cdot\ell_{3,4}) \;\to\; \frac{1}{(p_3\cdot\ell_{4,3})}\left[\beta\,p_3^\lambda D_\lambda - \gamma(q\cdot p_3)\right]\,, \tag{51}$$

from which the desired result follows (using Eqs. (10) and (19))

$$
\begin{aligned}
I_{3;\,\mu}^{(n)} &= \frac{\beta}{\gamma}\, J_{3;\,\mu}^{(n)} \;+\; \frac{1}{4}\left[\frac{\ell_{3\mu}}{(p_3\cdot\ell_3)}+\frac{\ell_{4\mu}}{(p_3\cdot\ell_4)}\right] \\
&\times \;\left\{f_{30} I_3^{(n)} + I_2^{(n)}(3) - I_2^{(n)}(0) - \frac{2\beta}{\gamma} p_3^{\lambda}\, J_{3;\,\lambda}^{(n)}\right\}. \qquad (52)
\end{aligned}
$$

5.4 The $m > 3$ case

Contracting Eq. (18) with $p_{3,4}$ one can write q_μ as a function of $(q\cdot p_i)$, $i = 1, \cdots, 4$:

$$
\begin{aligned}
q_\mu &= \frac{\beta}{\gamma} D_\mu + \frac{\ell_{3\mu}\ell_{4\alpha} - \ell_{3\alpha}\ell_{4\mu}}{2\,\delta}\left\{p_3^{\alpha}\left(2\,(q\cdot p_4) - \frac{2\beta}{\gamma} p_{4\lambda} D^{\lambda}\right) - p_4^{\alpha}\left(2\,(q\cdot p_3) - \frac{2\beta}{\gamma} p_{3\lambda} D^{\lambda}\right)\right\}, \\
\delta &= (\ell_3\cdot p_4)(\ell_4\cdot p_3) - (\ell_3\cdot p_3)(\ell_4\cdot p_4)\,. \qquad (53)
\end{aligned}
$$

Then, after reconstructing the denominators and integrating, one gets

$$
\begin{aligned}
I_{m;\,\mu}^{(n)} &= \frac{\beta}{\gamma} J_{m;\,\mu}^{(n)} + \frac{\ell_{3\mu}\ell_{4\alpha} - \ell_{3\alpha}\ell_{4\mu}}{2\,\delta}\left\{p_3^{\alpha}\left[f_{40} I_m^{(n)} + I_{m-1}^{(n)}(4) - I_{m-1}^{(n)}(0) - \frac{2\beta}{\gamma} p_{4\lambda}\, J_m^{(n)\,\lambda}\right]\right. \\
&- \left. p_4^{\alpha}\left[f_{30} I_m^{(n)} + I_{m-1}^{(n)}(3) - I_{m-1}^{(n)}(0) - \frac{2\beta}{\gamma} p_{3\lambda}\, J_m^{(n)\,\lambda}\right]\right\}. \qquad (54)
\end{aligned}
$$

Note that $|\delta|^2$ is proportional to the Gram determinant of the 4-momenta ℓ_1, ℓ_2, p_3, p_4. The appearance of inverse square roots of Gram determinants is a peculiarity of our formalism and it will be discussed at length in Section 6.

We close this Section by observing that Eq. (53) could be used instead of Eq. (24) when $m > 3$ to derive a recursion relation alternative to Eq. (9). In fact nothing prevents from multiplying by an arbitrary number of 4-vectors $q_\rho \cdots q_\tau$ before integrating. The reason why we prefer Eq. (24) is because it involves only three out of the m external 4-momenta, while a fourth momentum is necessary to write down Eq. (53). This has important consequences when studying collinear or coplanar configurations, as we shall see in next Section.

6 Study of exceptional configurations

We are interested in the behaviour of our formulae at the edges of the phase-space, where two or more momenta can become nearly linearly dependent. We shall first show that only square roots of the two Gram determinants

$$
\begin{aligned}
\Delta &= -\Delta_{12} = (p_1\cdot p_2)^2 - p_1^2 p_2^2 \\
\Delta_{123} &= 2\,(\ell_1\cdot\ell_2)(\ell_1\cdot p_3)(\ell_2\cdot p_3) - p_3^2\,(\ell_1\cdot\ell_2)^2 \qquad (55)
\end{aligned}
$$

appear in the denominators of Eq. (9). Furthermore, the only occurrence of a square root of a rank four Gram determinant is in Eq. (53). These two facts make our approach numerically more stable than conventional methods, in which the scalar coefficients of the tensor decomposition develop poles proportional to $1/\Delta_{12}$, $1/\Delta_{123}$ or $1/|\delta|^2$ at each step of the reduction. Secondly, we shall argue that Eq. (9) is such that the numerical cancellations occurring among the tensor loop functions in the limit of exceptional momenta are kept as local as possible. Otherwise stated, each of the three terms of Eq. (9) is separately well behaved when $\Delta_{12} \to 0$ or $\Delta_{123} \to 0$. Although this cannot solve by itself all problems of numerical inaccuracy, it helps in decreasing the values of Δ_{12} and Δ_{123} for which approximations to Eq. (9) should be used. Thirdly, we shall show how to deal with configurations with *exactly* zero Gram determinants and give general prescriptions on how to cure the numerical instabilities occurring near the zeros of Δ_{12} and Δ_{123}.

We should then investigate all possible denominators appearing in Eq. (9) in the limit of exceptional momenta. We start considering the 4-vectors $\ell_{1,2}$ in Eq. (14). Inserting $\alpha_{1,2}$ in the definition of β, one finds

$$\beta = \pm \frac{p_1^2}{2\alpha_1\sqrt{\Delta}}\,, \tag{56}$$

where the sign depends on the sign in α_1 (see Footnote 2). Despite of this, $\ell_{1,2}$ remain well behaved in the limit $\Delta_{12} \to 0$. In fact, splitting the external momentum p_2 as follows

$$p_2 = \eta p_1 + (p_2 - \eta p_1) \equiv \eta p_1 + \phi \hat{n}\,, \tag{57}$$

and choosing η and ϕ such that $(p_1 \cdot \hat{n}) = 0$ and $\hat{n}^2 = -p_1^2$, one gets

$$p_2 = \eta p_1 + \frac{\sqrt{\Delta}}{p_1^2}\,\hat{n}\,, \tag{58}$$

with

$$\eta = \frac{(p_1 \cdot p_2)}{p_1^2} \quad \text{and} \quad \hat{n}^\mu = \frac{1}{\sqrt{\Delta}}\left[p_1^2\, p_2^\mu - (p_1 \cdot p_2)\, p_1^\mu\right]. \tag{59}$$

Therefore, in terms of p_1 and $\hat{n}$, we get well defined expressions even when $\Delta_{12} \to 0$:

$$\ell_1 = \frac{1}{2}(p_1 \mp \hat{n})\,, \qquad \ell_2 = \frac{1}{2\alpha_1}(p_1 \pm \hat{n})\,. \tag{60}$$

If needed, one can use higher numerical accuracy just in the computation of $\hat{n}^\mu$, or choose a particular frame to stabilise the result. For example, for time-like p_1, one takes $\hat{n}^\mu = M\,(0, \vec{p}_2/|\vec{p}_2|)$ when $p_1^\mu = (M, \vec{0})$. Finally, when both p_i^2 vanish, $\alpha_{1,2} = 0$ as noticed in Footnote 2, and $\ell_{1,2} \equiv p_{1,2}$. When only one p_i^2 is zero, $\ell_{1,2}$ are still well defined provided $(p_1 \cdot p_2) \neq 0$, but this can also be cured, as discussed later.

Let us now analyse the terms β, $1/\gamma$, $1/(\ell_3 \cdot p_3)$ and $1/(\ell_4 \cdot p_3)$ in Eq. (9) in turn, where the last two quantities are hidden in the definition of the rank three tensor $T_{\mu\nu\lambda}$. We start considering the case $p_i^2 \neq 0$. Then β is proportional to $1/\sqrt{\Delta}$, but the combination of tensor loop functions multiplying β in Eq. (9) is always such that the product is well behaved in the limit $\Delta_{12} \to 0$. The proof is simple and follows from the fact that these terms come from D^μ defined in Eq. (18). Indeed, it can be shown using Eq. (60) that

$$D^\mu = \mp 2\frac{\sqrt{\Delta}}{p_1^2}\left[(q\cdot\hat{n})\hat{n}^\mu - (q\cdot p_1)p_1^\mu\right], \tag{61}$$

thus compensating the $1/\sqrt{\Delta}$ pole coming from β. Of course, we cannot simplify this pole analytically, because by doing so we cannot express D^μ back in terms of denominators. However, Eq. (61) shows that well behaved combinations of loop functions naturally arise in our method. This is in contrast with the well behaved groupings of Ref. [17], result of a very complicated compensation of logarithms and di-logarithms in groupings obtained by differentiating with respect to external parameters or by developing scalar integrals in $6+\epsilon$ or higher dimensions. Obviously, such a compensation has also to occur *after* integrating over d^4q. The nice feature of our approach is that this works *before* performing the actual integration, what keeps things much simpler. In addition, in our case the cancellations in the numerator should compensate a factor $1/\sqrt{\Delta}$, and not $1/\Delta$. Finally, when at least one p_i is massless, β is simply 1.

Next we concentrate on the zeros of γ. Since

$$\gamma = \frac{p_1^2 p_2^2}{(p_1\cdot p_2)\mp\sqrt{\Delta}}, \tag{62}$$

there is no problem for massive $p_{1,2}$. On the other hand, when $p_{1,2}$ are both massless, $\gamma = 2(p_1\cdot p_2)$ can vanish. However, such configurations correspond to true collinear singularities of the amplitude, which are cut away in physical observables. When $p_1^2 = 0$ and $p_2^2 \neq 0$ ($p_1^2 \neq 0$ and $p_2^2 = 0$), γ can become zero at the edges of the phase-space because $\gamma = 2(p_1\cdot p_2) = 0$ is a zero of the Gram determinant $\Delta_{p_1-p_2,p_2}$ (Δ_{p_1,p_2-p_1}). For loop functions with $m \geq 3$, one simply renames $p_{1,3}$ assigning the massless 4-momentum to p_3. On the other hand, if $m = 2$ and to fix things $p_1^2 = 0$, one redefines $\ell_{1,2}$ in Eq. (13) using $p_1 - p_2$ and p_2 instead of p_1 and p_2 in order to move this pole to β. The latter is important because as we have seen, β behaves as the inverse square root of a Gram determinant, while poles in $1/\gamma^2$ are present everywhere in our formulae. The net effect of this choice for $\ell_{1,2}$ is the replacement

$$I^\sigma_{m-1;\rho\cdots\tau}(1) \to I^\sigma_{m-1;\rho\cdots\tau}(1) - I^\sigma_{m-1;\rho\cdots\tau}(2) \tag{63}$$

in Eq. (9).

We turn to $1/(\ell_3\cdot p_3)$ and $1/(\ell_4\cdot p_3)$. They are both proportional to $1/\sqrt{\Delta_{123}}$. In fact,

$$\begin{aligned}|(\ell_4\cdot p_3)|^2 &= |(\ell_3\cdot p_3)|^2 = \bar{v}(\ell_1)\,\not{p}_3\,\omega^-\,u(\ell_2)\,\bar{v}(\ell_2)\,\not{p}_3\,\omega^-\,u(\ell_1) = \bar{v}(\ell_1)\,\not{p}_3\,\not{\ell}_2\not{p}_3\,\omega^-\,u(\ell_1)\\ &= \frac{1}{2}\mathrm{Tr}[\not{\ell}_1\not{p}_3\not{\ell}_2\not{p}_3] = \frac{2}{(\ell_1\cdot\ell_2)}\Delta_{123}\,. \end{aligned} \tag{64}$$

This completes the proof of that only square roots of Gram determinants appear at each step of our reduction procedure.

Now, we shall show that also the last two terms of Eq. (9) contain only well behaved combinations of loop functions in the limits $\Delta_{12} \to 0$ and $\Delta_{123} \to 0$. To prove this, we split the 4-vector p_3 in a similar way as we did with p_2 in Eq. (57)

$$p_3 = \eta_1 \ell_1 + \eta_2 \ell_2 + \phi \hat{m} \,. \tag{65}$$

Choosing η_1, η_2 and ϕ such that $(\ell_1 \cdot \hat{m}) = (\ell_2 \cdot \hat{m}) = 0$ and $\hat{m}^2 = -(\ell_1 \cdot \ell_2)^2$, one gets

$$\begin{aligned} \eta_1 &= \frac{(\ell_2 \cdot p_3)}{(\ell_1 \cdot \ell_2)} \,, \quad \eta_2 = \frac{(\ell_1 \cdot p_3)}{(\ell_1 \cdot \ell_2)} \,, \quad \phi = \frac{\sqrt{\Delta_{123}}}{(\ell_1 \cdot \ell_2)^2} \,, \\ \hat{m}^\mu &= \frac{(\ell_1 \cdot \ell_2)}{\sqrt{\Delta_{123}}} \left[(\ell_1 \cdot \ell_2)\, {p_3}^\mu - (\ell_2 \cdot p_3)\, {\ell_1}^\mu - (\ell_1 \cdot p_3)\, {\ell_2}^\mu \right] . \end{aligned} \tag{66}$$

Then

$$(\ell_{3,4} \cdot p_3) = \frac{\sqrt{\Delta_{123}}}{(\ell_1 \cdot \ell_2)^2} (\ell_{3,4} \cdot \hat{m}) \,, \tag{67}$$

implying that $T_{\mu\nu}$, defined in Eq. (23) and depending only on the ratio between $(\ell_3 \cdot p_3)$ and $(\ell_4 \cdot p_3)$, behaves smoothly in the limit $\Delta_{123} \to 0$. This tells us that the only singularity in the second term of Eq. (9) can come from $1/\gamma$, but this can be cured as explained above. To conclude, the last term of Eq. (9) is proportional to $T_{\mu\nu\lambda} \sim \frac{1}{\sqrt{\Delta_{123}}}$. Expressing $\ell_{1,2}$ in Eq. (65) in terms of $p_{1,2}$ one obtains

$$p_3 = \left[\frac{2\beta}{\gamma} (r_2 \cdot p_3) \right] p_1 + \left[\frac{2\beta}{\gamma} (r_1 \cdot p_3) \right] p_2 + \frac{\sqrt{\Delta_{123}}}{(\ell_1 \cdot \ell_2)^2} \, \hat{m} \,. \tag{68}$$

Then, using Eq. (68) the coefficient of $q^\lambda \, T_{\mu\nu\lambda}$ in Eq. (24) can be written as

$$-\frac{1}{2\gamma} (q \cdot \hat{m}) \, \frac{\sqrt{\Delta_{123}}}{(\ell_1 \cdot \ell_2)^2} \,. \tag{69}$$

Therefore, the last combination of loop functions in Eq. (9) must combine in such a way that the pole $1/\sqrt{\Delta_{123}}$ coming from the tensor gets compensated.

When Δ_{12} or Δ_{123} are *exactly* zero one has to rely on a different strategy. Let us concentrate on the case when, for example, $p_2 = \lambda p_1$ *exactly.* If m is large enough, one simply chooses within the set $\{p_j\}$ a different subset of three independent 4-momenta to perform the reduction. $\bar{D}_2$ acts then as a *spectator* denominator. When due to cancellations of denominators one is left with tensors integrals

$$\int d^n \bar{q} \frac{q_\mu \cdots q_\nu}{\bar{D}_0 \bar{D}_1 \bar{D}_2 \bar{D}_3} \,, \tag{70}$$

one switches to the reduction valid for 3-point functions. In fact, when $p_2 = \lambda p_1$,

$$\int d^n \bar{q} \frac{q_\mu \cdots q_\nu}{\bar{D}_0 \bar{D}_1 \bar{D}_2 \bar{D}_3} \quad \text{and} \quad \int d^n \bar{q} \frac{q_\mu \cdots q_\nu}{\bar{D}_0 \bar{D}_1 \bar{D}_3} \tag{71}$$

share the same tensor basis built up in terms of $p_{1,3}$ and metric tensors. At the end, only 2-point like tensors remain to be further reduced with the standard techniques of Ref. [2]. The same procedure also works when three or more momenta become linearly dependent. With the outlined method, reliable approximations can be easily obtained near the zeros of the Gram determinants, where numerical cancellations occur among the loop functions of each well behaved combination. When, for example, $\Delta_{12} \ll 1$, one simply starts the reduction using $\bar{D}_2 = (\bar{q}+\lambda p_1)^2 - m_2{}^2$. This is completely equivalent to the Taylor expansion performed in Ref. [17] to extract the constant terms in each grouping. The advantage of our approach is that we do not need to explicitly develop tensor functions. Just at the end, when everything is reduced to scalar $(4+\epsilon)$-dimensional loop integrals, only these need to be computed precisely, also in the regime of vanishing Gram determinants. This can be done, for example, as explained in Refs. [4, 8, 9, 17].

A problem which remains to be solved is the determination of the maximal values of Δ_{12}, Δ_{123} and δ in Eq. (53) below which the approximations to Eq. (9) should be used. As in Ref. [17] such values can be only found performing dedicated numerical studies, and it is then difficult to give general prescriptions. We do not want to get deeply involved into the subject here, but just mention that by taking advantage of the fact that our reduction procedure takes place *before* integration, tests on the numerical stability of the formalism are possible without even evaluating the loop integrals. For example, for any given arbitrary 4-vector q the *integrands* on the r.h.s. of Eq. (9) should add up in such a way that at the end of the recursive algorithm the result is numerically equivalent to

$$\frac{q_\mu q_\nu q_\rho \cdots q_\tau}{D_0 D_1 \cdots D_m} \,. \tag{72}$$

We have performed such a check on tensor integrals up to rank 4.

7 Summary

We have presented a method to compute numerically and recursively tensor integrals appearing in one-loop calculations, and relevant for the next generation of pp and e^+e^- colliders. The treatment is applicable irrespective of the number of external legs to any configuration of internal and/or external variables, and only requires the knowledge of the standard set of scalar one-loop integrals. We distinguish the cases of 3-point tensor integrals (Section 4), as well as of rank 1 (Section 5), which are treated separately with similar techniques to those used in the general case (Sections 2 and 3). Singular kinematical configurations are analised in detail (Section 6), finding a smoother behaviour than in other approaches. In addition, we have studied all possible sources of numerical instabilities, giving general prescriptions on how to cure them. A code implementing the proposed method will be made available in the near future.

Acknowledgments

R. P. would like to thank all members of the Department of Theoretical Physics of the University of Granada for the warm hospitality. This work is supported by the European Union Community's Human Potential Programme under contract HPRN-CT-2000-00149 Physics at Colliders, and by MECD under contact SAB2002-0207.

A The vectors $\ell_{3,4}$

The 4-vectors $\ell_{3,4}$ defined in Eq. (15) enjoy useful properties. By using the Dirac equation and the completeness relations for massless spinors one immediately derives

$$\begin{aligned}
(\ell_{3,4}\cdot\ell_{1,2}) &= 0\,,\\
\ell_3^2 &= \bar v(\ell_1)\,\gamma^\mu\,\omega^-\,u(\ell_2)\,\bar v(\ell_1)\,\gamma_\mu\,\omega^-\,u(\ell_2)\\
&= \frac{1}{\bar v(\ell_2)\,\not{b}\,u(\ell_1)}\,\bar v(\ell_1)\,\gamma^\mu\,\omega^-\,u(\ell_2)\,\bar v(\ell_2)\,\not{b}\,u(\ell_1)\,\bar v(\ell_1)\,\gamma_\mu\,\omega^-\,u(\ell_2)\\
&= \frac{1}{\bar v(\ell_2)\,\not{b}\,u(\ell_1)}\,\bar v(\ell_1)\,\gamma^\mu\,\not{\ell}_2\,\not{b}\,\not{\ell}_1\,\gamma_\mu\,\omega^-\,u(\ell_2)\\
&= \frac{-2}{\bar v(\ell_2)\,\not{b}\,u(\ell_1)}\,\bar v(\ell_1)\,\not{\ell}_1\,\not{b}\,\not{\ell}_2\,\omega^-\,u(\ell_2) \;=\; 0\,,\\
\ell_4^2 &= \bar v(\ell_2)\,\gamma^\mu\,\omega^-\,u(\ell_1)\bar v(\ell_2)\,\gamma_\mu\,\omega^-\,u(\ell_1) \;=\; 0\,,
\end{aligned} \tag{73}$$

where b is an arbitrary 4-vector different from $\ell_{1,2}$. Furthermore,

$$\begin{aligned}
(\ell_3\cdot\ell_4) &= \bar v(\ell_1)\,\gamma_\mu\,\omega^-\,u(\ell_2)\,\bar v(\ell_2)\,\gamma^\mu\,\omega^-\,u(\ell_1) \;=\; \bar v(\ell_1)\,\gamma_\mu\,\not{\ell}_2\gamma^\mu\,\omega^-\,u(\ell_1)\\
&= \frac{1}{2}\mathrm{Tr}[\not{\ell}_1\gamma_\mu\not{\ell}_2\gamma^\mu] \;=\; -\,4\,(\ell_1\cdot\ell_2)\,,\\
(q\cdot\ell_3)(q\cdot\ell_4) &= \bar v(\ell_1)\,\not{q}\,\omega^-\,u(\ell_2)\,\bar v(\ell_2)\,\not{q}\,\omega^-\,u(\ell_1) \;=\; \bar v(\ell_1)\,\not{q}\,\not{\ell}_2\not{q}\,\omega^-\,u(\ell_1)\\
&= \frac{1}{2}\mathrm{Tr}[\not{\ell}_1\not{q}\not{\ell}_2\not{q}] \;=\; 4\,(q\cdot\ell_1)(q\cdot\ell_2) - 2\,q^2\,(\ell_1\cdot\ell_2)\,,\\
(q\cdot\ell_3)(q\cdot\ell_3) &= \frac{1}{(b\cdot\ell_4)}\bar v(\ell_1)\,\not{q}\,\not{\ell}_2\;\not{b}\,\not{\ell}_1\;\not{q}\,\omega^-\,u(\ell_2)\\
&= \frac{2}{(b\cdot\ell_4)}\Big\{[q^2(\ell_1\cdot\ell_2) - 2(q\cdot\ell_1)(q\cdot\ell_2)](b\cdot\ell_3)\\
&\quad +2\,[(q\cdot\ell_1)(b\cdot\ell_2) - (q\cdot b)(\ell_1\cdot\ell_2) + (q\cdot\ell_2)(\ell_1\cdot b)](q\cdot\ell_3)\Big\}\,.
\end{aligned} \tag{74}$$

In the same way one computes $(q\cdot\ell_4)(q\cdot\ell_4)$.

For numerical applications one needs $\ell_{3,4}$ in terms of $\ell_{1,2}$. Given $\ell_i^\mu = (\ell_{i0},\ell_{ix},\ell_{iy},\ell_{iz})$ and using [25]:

$$\sigma^0 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad \sigma^1 = \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix},\quad \sigma^2 = \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix},\quad \sigma^3 = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix},$$

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & -\sigma^k \\ \sigma^k & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix}, \tag{75}$$

$$\bar{v}(\ell_i)\,\omega^+ = (b_i, c_i^-, 0, 0), \quad \omega^- u(\ell_i) = \begin{pmatrix} 0 \\ 0 \\ b_i \\ c_i^+ \end{pmatrix}, \quad b_i = \sqrt{\ell_{i0} + \ell_{iz}}, \quad c_i^\pm = \frac{\ell_{ix} \pm i\ell_{iy}}{b_i},$$

the desired result follows

$$\begin{array}{ll} \ell_{30} = b_1 b_2 + c_1^- c_2^+ , & \ell_{40} = b_2 b_1 + c_2^- c_1^+ , \\ \ell_{3x} = b_1 c_2^+ + c_1^- b_2 , & \ell_{4x} = b_2 c_1^+ + c_2^- b_1 , \\ \ell_{3y} = i(c_1^- b_2 - b_1 c_2^+) , & \ell_{4y} = i(c_2^- b_1 - b_2 c_1^+) , \\ \ell_{3z} = b_1 b_2 - c_1^- c_2^+ , & \ell_{4z} = b_2 b_1 - c_2^- c_1^+ . \end{array} \tag{76}$$

B Extra integrals

In this Appendix we compute the extra integrals appearing in the n-dimensional version of the proposed reduction method

$$I^{(n;\,2\ell)}_{m;\,\mu_1\cdots\mu_{2s}} = \int d^n\bar{q}\ \tilde{q}^{2\ell}\ \frac{q_{\mu_1}\cdots q_{\mu_{2s}}}{\bar{D}_0\cdots\bar{D}_m}, \tag{77}$$

where $\ell > 0$ and $2s$ are non-negative integers. It is convenient to define a new index $d = \ell + s + 1 - m$ to classify the integrals according to their dimensionality in powers of $[mass]^2$. One can convince him/herself that tensor integrals have a non-zero contribution at $\mathcal{O}(1)$, which is the order we are interested in, only when $d \geq 0$. For example,

$$I^{(n;\,2)}_{2;\,\mu} = \frac{i\pi^2}{6}(p_{1\mu} + p_{2\mu}) + \mathcal{O}(\epsilon). \tag{78}$$

Their general expression can be easily written for $d \leq 0$:

$$I^{(n;\,2(m-1+d-s))}_{m;\,\mu_1\cdots\mu_{2s}} = \begin{cases} -i\pi^2\ \frac{g_{\mu_1\cdots\mu_{2s}}}{2^s}\ \frac{\Gamma(m-s-1)}{\Gamma(m+1)} + \mathcal{O}(\epsilon) & \text{when } d = 0\ , \\ 0 & \text{when } d < 0\ , \end{cases} \tag{79}$$

where the symmetric combination $g_{\mu_1\cdots\mu_{2s}}$ is as in Eq. (29), but for 4-dimensional metric tensors.

Let us discuss the scalar case ($s = 0$) in more detail. Decomposing the integration [26]

$$\int d^n\bar{q} = \int d^4q\, d^\epsilon\mu \qquad (\tilde{q}^2 = -\mu^2) \tag{80}$$

and after using Feynman parametrization, one gets

$$\begin{aligned} I^{(n;\,2(m-1+d))}_{m} &= (-)^{2m+d}\, i\pi^2 \frac{\pi^{\epsilon/2}}{\Gamma(\epsilon/2)}\Gamma\left(m-1+d+\frac{\epsilon}{2}\right)\Gamma\left(-d-\frac{\epsilon}{2}\right) \\ &\times \int [d\alpha]_m\ \mathcal{X}_m^{(d+\frac{\epsilon}{2})}, \end{aligned} \tag{81}$$

with

$$\int [d\alpha]_m = \int_0^\infty d\alpha_0 \cdots d\alpha_m \; \delta(1 - \sum_{k=0}^{m} \alpha_k)\,, \quad \mathcal{X}_m = P_m^2 + M_m^2$$

$$P_m = \sum_{k=1}^{m} \alpha_k p_k\,, \quad M_m^2 = \alpha_0 m_0^2 + \sum_{k=1}^{m} \alpha_k (m_k^2 - p_k^2)\,. \tag{82}$$

We explicitly compute here one-loop integrals coming from tensors with rank at most equal to the number of denominators. This is enough for most practical calculations and gives rise to only one possibility with $d = 1$, namely $m = 1$. A straightforward calculation for this integral, gives

$$I_1^{(n;\,2)} \equiv \int d^n\bar{q}\, \frac{\tilde{q}^2}{\bar{D}_0 \bar{D}_1} = -i\frac{\pi^2}{2}\left[m_1^2 + m_0^2 - \frac{p_1^2}{3}\right] + \mathcal{O}(\epsilon)\,. \tag{83}$$

When $m_0^2 = m_1^2 = p_1^2 = 0$ the integral vanishes in dimensional regularization.

Furthermore, all scalar integrals with $d = 0$ are easily evaluated by observing that $\int [d\alpha]_m = 1/\Gamma(m+1)$:

$$I_m^{(n;\,2(m-1))} = -i\pi^2 \frac{\Gamma(m-1)}{\Gamma(m+1)} + \mathcal{O}(\epsilon) \quad \text{when} \;\; d = 0\,. \tag{84}$$

When $d = -1$, Eq. (81) gives

$$I_m^{(n;\,2(m-2))} = -i\pi^2 \frac{\pi^{\epsilon/2}}{\Gamma(\epsilon/2)} \Gamma\left(m - 2 + \frac{\epsilon}{2}\right) \Gamma\left(1 - \frac{\epsilon}{2}\right) \times \int [d\alpha]_m \; \mathcal{X}_m^{(-1+\frac{\epsilon}{2})}\,. \tag{85}$$

Since the constraint $\ell > 0$ in Eq. (77) implies $m - 2 > 0$, one could conclude that

$$I_m^{(n;\,2(m-2))} = 0 + \mathcal{O}(\epsilon) \tag{86}$$

if and only if the integral over the Feynman parameters in the second line of (85) does not contains poles in $1/\epsilon$. One may wonder whether this can actually occur for there may be infrared and collinear divergences. A simple reasoning shows that this is never the case. In fact, when $\ell > 0$, the presence of $\tilde{q}^{2\ell}$ in the numerator always raises the powers of the quadratic form $\mathcal{X}_m$ in the Feynman parameter integrals with respect to the "standard" $\ell = 0$ loop functions, therefore forbidding the presence of collinear or soft divergences. The same reasoning shows that

$$I_m^{(n;\,2(m-1+d))} = 0 + \mathcal{O}(\epsilon) \quad \text{when} \;\; d < 0\,. \tag{87}$$

However, it is extremely instructive to compute Eq. (85) directly. For the most collinear and infrared divergent fully massless box diagram in Fig. B one gets the finite expression

$$\int [d\alpha]_3 \, \frac{1}{\mathcal{X}_3} = -\frac{1}{s+t}\left[\ln(s/t)\ln(-s/t) + \mathrm{Li}_2(1 + t/s) + \mathrm{Li}_2(1 + s/t)\right]\,. \tag{88}$$

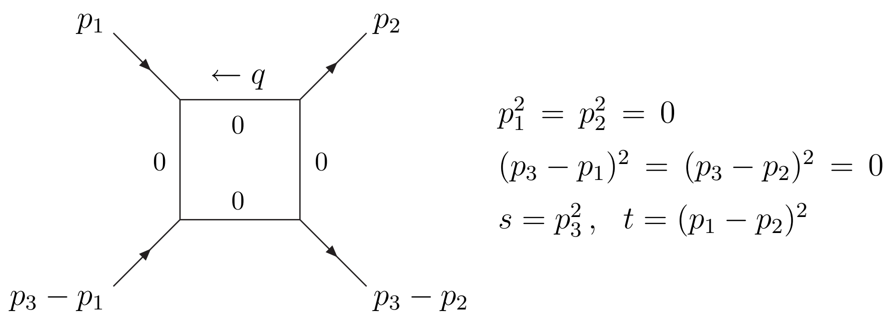

Fig. B: *Fully massless box diagram.*

Since all higher point loop functions can be rewritten as combinations of box diagrams [4, 7], this result explicitly proves Eq. (87).

We close this Appendix noting that, in any case, a contribution $\mathcal{O}(1)$ can only develop for non-negative powers of the quadratic form $\mathcal{X}_m$, so that the integrand in the Feynman parameter integral is always polynomial.

C Three-point tensors: general case

This Appendix extends Eq. (35) to three-point tensor integrals of rank higher than 3. We redefine

$$d_\mu \equiv \frac{\beta}{\gamma} D_\mu \quad \text{and} \quad L_\mu \equiv -\frac{1}{2\gamma} Q_\mu \,, \tag{89}$$

so that the first line of Eq. (18) simply reads $q_\mu = d_\mu + L_\mu$. Then

$$q_{\mu_1} \cdots q_{\mu_i} = \sum_{k=1}^{i} \left\{ L_{\mu_1} \cdots L_{\mu_{k-1}} \, d_{\mu_k} \, q_{\mu_{k+1}} \cdots q_{\mu_i} \right\} + \prod_{k=1}^{i} L_{\mu_k} \,. \tag{90}$$

The first term always contains reconstructed denominators and we do not elaborate it any further. Inserting the definition of Q_μ in Eq. (18) in the second term this can be written

$$\prod_{k=1}^{i} L_{\mu_k} = \left(-\frac{1}{2\gamma} \right)^i \sum_{k=0}^{i} (q \cdot \ell_3)^k (q \cdot \ell_4)^{(i-k)} \, S^k_{\mu_1 \cdots \mu_i} \,, \tag{91}$$

where the tensor $S^k_{\mu_1 \cdots \mu_i}$ is defined to be the sum of all possible tensor products of i 4-vectors ℓ_3 and ℓ_4, such that ℓ_4 appears k times. For example,

$$S^2_{\mu_1 \mu_2 \mu_3} = \ell_{4\mu_1} \ell_{4\mu_2} \ell_{3\mu_3} + \ell_{4\mu_1} \ell_{3\mu_2} \ell_{4\mu_3} + \ell_{3\mu_1} \ell_{4\mu_2} \ell_{4\mu_3} \,. \tag{92}$$

The two terms with $k = 0$ and $k = i$ in Eq. (91) do not contribute to the integral, due to the Theorem in Eq. (37). The product $(q \cdot \ell_3)(q \cdot \ell_4) = \beta q^\lambda D_\lambda - \gamma q^2$ can be factorised out from all the remaining terms, giving as result

$$\prod_{k=1}^{i} L_{\mu_k} = \left(-\frac{1}{2\gamma}\right)^i [\beta q^\lambda D_\lambda - \gamma q^2]\, q^{\alpha_1} \cdots q^{\alpha_{i-2}}\, S_{[\alpha_1\cdots\alpha_{i-2}]\,\mu_1\cdots\mu_i}$$

$$S_{[\alpha_1\cdots\alpha_{i-2}]\,\mu_1\cdots\mu_i} \equiv \sum_{k=1}^{i-1} S^{i-k-1}_{\alpha_1\cdots\alpha_{i-k-1}}\, S^0_{\alpha_{i-k}\cdots\alpha_{i-2}}\, S^k_{\mu_1\cdots\mu_i}\,. \tag{93}$$

Inserting Eq. (93) into Eq. (90), dividing by the three denominators and integrating over $d^n q$ we obtain the desired relation

$$\begin{aligned} I^{(n)}_{2;\,\mu_1\cdots\mu_i} &= \frac{\beta}{\gamma}\sum_{k=1}^{i}\left(-\frac{1}{2\gamma}\right)^{k-1} t^{\alpha_2}_{\mu_1}\cdots t^{\alpha_k}_{\mu_{k-1}}\left\{J^{(n)}_{2;\,\mu_k\alpha_2\cdots\alpha_k\mu_{k+1}\cdots\mu_i}\right\} + \left(-\frac{1}{2\gamma}\right)^i S^{[\alpha_1\cdots\alpha_{i-2}]}_{\mu_1\cdots\mu_i} \\ &\times \left\{\beta J^{(n)\lambda}_{2;\,\lambda\alpha_1\cdots\alpha_{i-2}} - \gamma\left[m_0^2\, I^{(n)}_{2;\,\alpha_1\cdots\alpha_{i-2}} + I^{(n)}_{1;\,\alpha_1\cdots\alpha_{i-2}}(0) - I^{(n;\,2)}_{2;\,\alpha_1\cdots\alpha_{i-2}}\right]\right\}\,, \end{aligned} \tag{94}$$

where $S^{[\alpha_1\cdots\alpha_{i-2}]}_{\mu_1\cdots\mu_i}$ is given in Eq. (93), but with the α indices lowered, and as in Eq. (35) the tensors in the numerator of the n-dimensional integrals are 4-dimensional.

References

[1] F. A. Berends and W. T. Giele, Nucl. Phys. B **306**, 759 (1988);
F. A. Berends, W. T. Giele and H. Kuijf, Phys. Lett. B **232**, 266 (1989);
F. A. Berends, H. Kuijf, B. Tausk and W. T. Giele, Nucl. Phys. B **357**, 32 (1991);
F. Caravaglios and M. Moretti, Phys. Lett. B **358**, 332 (1995) [arXiv:hep-ph/9507237];
P. Draggiotis, R. H. Kleiss and C. G. Papadopoulos, Phys. Lett. B **439**, 157 (1998) [arXiv:hep-ph/9807207];
P. D. Draggiotis, R. H. Kleiss and C. G. Papadopoulos, Eur. Phys. J. C **24**, 447 (2002) [arXiv:hep-ph/0202201];
M. L. Mangano, M. Moretti, F. Piccinini, R. Pittau and A. D. Polosa, JHEP **0307**, 001 (2003) [arXiv:hep-ph/0206293].

[2] G. Passarino and M. J. G. Veltman, Nucl. Phys. B **160** (1979) 151.

[3] G. 't Hooft and M. J. G. Veltman, Nucl. Phys. B **153** (1979) 365.

[4] O. V. Tarasov, Phys. Rev. D **54** (1996) 6479 [arXiv:hep-th/9606018].

[5] A. I. Davydychev, Phys. Lett. B **263** (1991) 107.

[6] W. T. Giele and E. W. N. Glover, arXiv:hep-ph/0402152.

[7] T. Binoth, J. P. Guillet and G. Heinrich, Nucl. Phys. B **572** (2000) 361 [arXiv:hep-ph/9911342].

[8] G. Duplancic and B. Nizic, arXiv:hep-ph/0303184.

[9] T. Binoth, G. Heinrich and N. Kauer, Nucl. Phys. B **654** (2003) 277 [arXiv:hep-ph/0210023].

[10] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B **435** (1995) 59 [arXiv:hep-ph/9409265].

[11] D. E. Soper, Phys. Rev. D **62** (2000) 014009 [arXiv:hep-ph/9910292];
Phys. Rev. D **64** (2001) 034018 [arXiv:hep-ph/0103262].

[12] Z. Nagy and D. E. Soper, JHEP **0309** (2003) 055 [arXiv:hep-ph/0308127].

[13] A. Ferroglia, M. Passera, G. Passarino and S. Uccirati, Nucl. Phys. B **650** (2003) 162 [arXiv:hep-ph/0209219].

[14] G. Devaraj and R. G. Stuart, Nucl. Phys. B **519** (1998) 483 [arXiv:hep-ph/9704308].

[15] R. G. Stuart and A. Gongora, Comput. Phys. Commun. **56** (1990) 337.

[16] R. G. Stuart, Comput. Phys. Commun. **48** (1988) 367.

[17] J. M. Campbell, E. W. N. Glover and D. J. Miller, Nucl. Phys. B **498** (1997) 397 [arXiv:hep-ph/9612413].

[18] W. Beenakker, S. Dittmaier, M. Kramer, B. Plumper, M. Spira and P. M. Zerwas, Phys. Rev. Lett. **87** (2001) 201805 [arXiv:hep-ph/0107081];
DESY 02-177, hep-ph/0211352.

[19] A. Denner and S. Dittmaier, Nucl. Phys. B **658** (2003) 175 [arXiv:hep-ph/0212259].

[20] R. Pittau, Comput. Phys. Commun. **111** (1998) 48 [arXiv:hep-ph/9712418];
Comput. Phys. Commun. **104** (1997) 23 [arXiv:hep-ph/9607309].

[21] Z. Bern and A. G. Morgan, Nucl. Phys. B **467** (1996) 479 [arXiv:hep-ph/9511336].

[22] Z. Bern and D. A. Kosower, Nucl. Phys. B **379** (1992) 451.

[23] M. J. G. Veltman, Nucl. Phys. B **319** (1989) 253.

[24] S. Dittmaier, Nucl. Phys. B **675** (2003) 447 [arXiv:hep-ph/0308246].

[25] J. G. M. Kuijf, "Multiparton production at hadron colliders," Ph.D. Thesis, RX-1335 (LEIDEN).

[26] G. Mahlon, Phys. Rev. D **49** (1994) 2197 [arXiv:hep-ph/9311213];
Phys. Rev. D **49** (1994) 4438 [arXiv:hep-ph/9312276].